Journal of Statistical Physics, Vol. 47, Nos. 5/6, 1987

Upper Bounds on Correlation Decay for One-Dimensional Long-Range Spin-Glass Models

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Received October 1, 1986

Some recent results on one-dimensional spin-glass models with polynomially decreasing interactions are described.

KEY WORDS: Long range; one-dimensional spin-glass; asymptotic correlation decay.

In this contribution we describe some recent results on the absence of phase transitions in one-dimensional long-range spin-glasses. The systems we consider have the following Hamiltonian:

$$H = \sum_{i,j} |i-j|^{-\alpha} J(i, j)\, s_i s_j \tag{1}$$

where the $J(i, j)$ are i.i.d., which satisfy

$$\mathbb{E} \exp tJ(i, j) = \exp[t^2 \mathbb{E}J(i, j)^2 + O(t^3)] \tag{2}$$

for small t, that is, they have zero average and a convergent cumulant expansion. The s_i can be either Ising or vector spins and we consider the case $\alpha > 1$ in one dimension. We prove some results that support the general idea that a random potential that decays like $|i-j|^{-\alpha}$ behaves in some sense like a nonrandom, effective potential that decays like $|i-j|^{-2\alpha}$, that is, the potential gets effectively squared. For other work supporting this idea see Refs. 3–12. We remind the reader that for nonrandom potentials there is no transition (in the strong sense of analyticity) for $\alpha > 2$,[22,23] while there is spontaneous magnetization at low temperatures in the

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0022-4715/87/0600-0905$05.00/0

ferromagnetic case if $1 < \alpha \leqslant 2$.[24,25] However, because of the occurrence of Griffiths singularities,[26] analyticity is not to be expected in our models.

We have obtained the following results, almost surely with respect to the $J(i, j)$.

The Gibbs state is pure and does not depend on (fixed) boundary conditions (for this "weak uniqueness" see Refs. 1, 3, and 4), and

$$|\langle s_0 s_j \rangle| \leqslant C(\{J\}) \, |j|^{-\delta} \tag{3}$$

for any $\delta < \alpha - 1$ in the case of Ising spins and $\delta < \alpha - 1/2$ for n-vector spins. The random constant $C(\{J\})$ is almost surely finite and does not depend on the distance j.

The conjectured best value for δ is $\delta = \alpha$. This has been proven to hold at high T in Ref. 4 for general dimension d and $\alpha > \frac{1}{2}d$.

We sketch here the main ideas of the proofs; for full details and some extensions see Refs. 1 and 2.

The first ingredient is to apply Fubini's theorem to *modified* thermal expectations. This often makes it possible to replace terms $|i-j|^{-\alpha}$ after averaging over the $\{J\}$ by terms $|i-j|^{-2\alpha}$.

Assume that we can split the Hamiltonian (1) and write it as

$$H = H_0 + V \tag{4a}$$

where

$$V(\{J\}, \{s\}) = \sum_{i,j}' |i-j|^{-\alpha} J(i, j) \, s_i s_j \tag{4b}$$

We do not specify here which terms are taken in the primed sum. Different choices for V are made at different steps in the proof. We only mention the fact that in general V is chosen such that for all spin configurations $\{s\}$

$$\mathbb{E}_J \exp V \approx \exp\left[\sum_{i,j}' |i-j|^{-2\alpha} \, \mathbb{E}J(i, j)^2\right] < \infty \tag{5}$$

At some stages we consider V's for which the above expression is close to unity (that is, V is small with high probability).

Let us define the "good" set

$$G_c = \{\{J\}, \{s\} \,|\, |V(\{J\}, \{s\}| \leqslant c\} \tag{6a}$$

and the "bad" set

$$B_c = \{\{J\}, \{s\} \,|\, |V(\{J\}, \{s\}| > c\} \tag{6b}$$

The constant c will be chosen large compared to $[\mathbb{E}(V^2)]^{1/2}$.

In the case where c is nevertheless small, we can write on the "good" set G_c for any positive observable f

$$\mathbb{E}\langle f\rangle_H=\mathbb{E}\left(\frac{\langle f \exp V\rangle_{H_0}}{\langle \exp V\rangle_{H_0}}\right)=\mathbb{E}\left(\frac{\langle f \exp V\rangle_{H_0}}{1+\langle \exp V-1\rangle_{H_0}}\right) \tag{7}$$

We can develop (7) in a Taylor expansion in V and then apply Fubini's theorem to interchange the average over the $J(i, j)$, which occur in V with the modified thermal expectation $\langle\cdots\rangle_{H_0}$. Since

$$\mathbb{E}\langle V\rangle_{H_0}=\langle \mathbb{E}V\rangle_{H_0}=0 \tag{8}$$

the leading term containing V is of order $\mathbb{E}V^2 \simeq \sum'_{i,j}|i-j|^{-2\alpha}$.

In the case where c is large, we use a different argument to apply on the "good" set.

For the "bad" set B_c, we will use condition (2). Under this condition the exponential Chebyshev (or Bernstein) inequality for $J(i, j)$ holds[5,13-16]:

$$\operatorname{Prob}(|J(i,j)|>2c)\leqslant 2\exp[-c^2/\mathbb{E}J(i,j)^2] \tag{9}$$

We again use modified thermal expectations and obtain

$$\begin{aligned}
\mathbb{E}\langle\chi_{B_c}\rangle_H &= \mathbb{E}\langle\chi_{B_c}\exp V\rangle_{H_0}/\langle\exp V\rangle_{H_0}\\
&\leqslant \mathbb{E}\langle\chi_{B_c}\exp V\rangle_{H_0}\exp(-\langle V\rangle_{H_0})\\
&\leqslant (\mathbb{E}\langle\chi_{B_c}\exp V\rangle^2_{H_0})^{1/2}(\mathbb{E}\exp -2\langle V\rangle_{H_0})^{1/2}\\
&\leqslant (\mathbb{E}\langle\chi_{B_c}\rangle_{H_0}\langle\exp 2V\rangle_{H_0})^{1/2}(\mathbb{E}\exp -2\langle V\rangle_{H_0})^{1/2}\\
&\leqslant (\mathbb{E}\langle\chi_{B_c}\rangle^2_{H_0})^{1/4}(\mathbb{E}\langle\exp 2V\rangle^2_{H_0})^{1/4}(\mathbb{E}\exp -2\langle V\rangle_{H_0})^{1/2}\\
&\leqslant (\mathbb{E}\langle\chi_{B_c}\rangle_{H_0})^{1/4}(\mathbb{E}\langle\exp 4V\rangle_{H_0})^{1/4}(\mathbb{E}\exp -2\langle V\rangle_{H_0})^{1/2}
\end{aligned} \tag{10}$$

The first term on the right-hand side of this inequality can be made small because of (9) and Fubini's theorem, and the other terms remain finite due to a condition like (5).

In deriving (10), we just used Jensen's inequality and the Cauchy–Schwarz inequality. For a different derivation of this result, see Ref. 1. Note that (10) is a Bernstein-like inequality for a quantity inside the thermal average.

The second ingredient in our proofs is an argument for the deterministic effective Hamiltonian.

In Ref. 2 we used the McBryan–Spencer inequality[16-21]

$$|\langle \mathbf{s}_0\mathbf{s}_j\rangle|\leqslant\exp[-(a_0-a_j)]\,Z(H')/Z(H) \tag{11a}$$

where the primed Hamiltonian H' is given by

$$J'(i, j) = \cosh(a_i - a_j)\, J(i, j) \tag{11b}$$

We choose the a_i as in Messager *et al.*[17] (see also Ref. 16):

$$a_{|j|} - a_{|j|-1} = K/|j| \tag{12}$$

and we can apply their estimates fairly straightforwardly to derive the inequality (3).

The argument in Ref. 1 is somewhat more complicated for the spin-glass case. Here we present it for the nonrandom case, where it is reasonably simple.

Proposition. Let $H = -\sum_{i,j \in \mathbb{Z}} |i-j|^{-\alpha} s_i s_j$ and $\alpha > 2$. Then asymptotically

$$\langle s_0 s_j \rangle \leqslant |j|^{-\delta} \tag{3'}$$

for any $\delta < \alpha - 2$.

Remark. In fact, in this case it is known[22,23] that asymptotically $\langle s_0 s_j \rangle \sim |j|^{-\alpha}$. However, the proof of this stronger result is more complicated and we do not know yet how to apply it to the spin-glass models (but see Ref. 14).

Sketch of Proof. Consider an interval $\Lambda = [-nN, nN+n]$, which is divided into $2N+1$ blocks of size n. Both the block size and the number of blocks are chosen dependent on Λ: $N = O(|\Lambda|^{\gamma})$ and $n = O(|\Lambda|^{1-\gamma})$.

Let us write

$$H = H_{0,\Lambda} + V_\Lambda + H_{\text{outside}}$$

where $H_{0,\Lambda} + H_{\text{outside}}$ contains all the terms within blocks in Λ, between nearest neighbor blocks in Λ, and between the end blocks and the outside of Λ. The term V_Λ contains all the rest of the terms with at least one site in Λ. Then

$$\begin{aligned}\|V_\Lambda\| &= \sum_{\text{block } k = -N}^{N} \sum_{\substack{i \in \text{block } k \\ j \in n\text{-distant blocks}}} |i-j|^{-\alpha} = O(Nn^{2-\alpha}) \\ &= O(|\Lambda|^{\gamma + (2-\alpha)(1-\gamma)}) = O(|\Lambda|^{-\delta})\end{aligned} \tag{13}$$

by choosing γ small enough.

Because $\|V_\Lambda\|$ is small, we can write

$$\langle s_0 s_j \rangle_H \leqslant \langle s_0 s_j \rangle_{H_{0,\Lambda} + H_{\text{outside}}} + O(\|V_\Lambda\|) \tag{14}$$

For $\langle s_0 s_j \rangle_{H_{0,\Lambda} + H_{\text{outside}}}$ we can apply a Markov chain or transfer matrix argument, which gives us exponential decay in the block distance for j up to $\frac{1}{2}|\Lambda|$:

$$\langle s_0 s_j \rangle_{H_{0,\Lambda} + H_{\text{outside}}} \lesssim \exp(-|j|/|\Lambda|^{1-\gamma}) \tag{15}$$

Asymptotically (13) dominates (15) for Λ large, and, combined with (14), this gives us the announced upper bound (3′).

ACKNOWLEDGMENTS

These results have been obtained together with M. Campanino and E. Olivieri. Some ideas used here were developed in former work with J. L. van Hemmen and J. Fröhlich. I want to thank them for pleasant collaborations and for everything they taught me about these problems. I am grateful to the IAMP for supporting my visit to Třeboň. I thank the organizers for a pleasant conference and in particular Dr. M. Zahradnik for his hospitality. The author is a Lady Davis Fellow.

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